Technical Comments

Comment on "Derivation of Axisymmetric Flexural Vibration Equations of a Cylindrically Aeolotropic Circular Plate"

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THOUGH the basic governing differential equations for a cylindrically aeolotropic plate subjected to bending and vibration have been developed by several researchers,[1–5] the axisymmetric flexural vibration equations and associated boundary conditions of a cylindrically aeolotropic circular plate of varying thickness have been ingeniously derived by Joung[6] using the method of variational calculus. As stated by Juong the advantage of this method is that both the differential equations and the appropriate boundary conditions can be obtained simultaneously.

In Joung's Note some errors should be noted and corrections made. As a result of the extremizations of the functional,

$$I = \int_{t_1}^{t_2}\iint_{s} f(t,r,w,w_r,w_t,w_{rr})rdrd\theta dt \quad (1)$$

the differential equation of motion and the appropriate boundary conditions should read

$$D(c^2w_{rrrr} + 2c^2r^{-1}w_{rrr} - r^{-2}w_{rr} + r^{-3}w_r) + D_r[2c^2w_{rrr} + (2c^2 + \nu)r^{-1}w_{rr} - r^{-2}w_r] + D_{rr}(c^2w_{rr} + \nu r^{-1}w_r) + \mu w_{tt} = p(r,t) \quad (2)$$

$$[rD(c^2w_{rr} + \nu r^{-1}w_r)\eta_r]_b{}^a = 0$$

and

$$\{[rD(c^2w_{rrr} + c^2r^{-1}w_{rr} - r^{-2}w_r) + rD_r(c^2w_{rr} + \nu r^{-1}w_r)]\eta\}_b{}^a = 0 \quad (3)$$

For isotropic material, $c^2 = 1$, Eqs. (2) and (3) yield the results given by Eqs. (10) through Eqs. (13) in the Note. However, for the case of an annular plate of aeolotropic material with constant thickness, the differential equation should read

$$c^2w_{rrrr} + 2c^2r^{-1}w_{rrr} - r^{-2}w_{rr} + r^{-3}w_r + D^{-1}\mu w_{tt} = D^{-1}p(r,t) \quad (4a)$$

and the boundary conditions for simple support and free hole are

$$w = 0, \quad c^2w_{rr} + \nu r^{-1}w_r = 0 \quad \text{at } r = a \quad (4b)$$

$$c^2w_{rr} + \nu r^{-1}w_r = 0$$

and

$$c^2w_{rrr} + c^2r^{-1}w_{rr} - r^{-2}w_r = 0 \quad \text{at } r = b \quad (4c)$$

For static problems, Eq. (4a) reduces to the differential equation for the axisymmetric bending of the cylindrically aeolotropic plate, which is given by Carrier as Eq. (7) in Ref. 1.

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Incidentally, a typographical error appears both in Ref. 1 and Ref. 6. The definition of ν should be

$$\nu = -a_{12}/a_{22}.$$

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Reply by Author to C. L. Huang

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Comment on "Separation Solutions for Laminar Boundary Layer"

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IN Ref. 1 Fox and Saland report on the separation solutions for the compressible similar laminar boundary-layer equations with mass transfer. The reported results were limited to unit Prandtl number.

In Ref. 1 the governing boundary-value problem for the compressible similar laminar boundary layer with mass transfer [Eqs. (1, 2, 5, and 6) of Ref. 1] was integrated using a

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Table 1 Comparison of separation eigenvalues for the similar compressible laminar boundary-layer equations without mass transfer
$\sigma = 1.0, f''(0) = 0, f(0) = 0$

g_w	$-\hat{\beta}$	$g'(0)$	$-\hat{\beta}$	$g'(0)$	$-\hat{\beta}$	$g'(0)$
	Fox and Saland[1]		Cohen and Reshotko[2]		Rogers[3]	
0.0	0.32718	0.24813	0.326	0.2477	0.326419	0.247790
0.2	0.30888	0.22618	0.3088	0.2261	0.308622	0.226014
0.6	0.24777	0.12518	0.2460	0.1249	0.247574	0.125103
1.0	0.19901	0	0.1988	0	0.198838	0
2.0	0.12962	−0.33913	0.1295	−0.3388	0.129507	−0.338911

trapezoidal rule with a step size of 0.1. The asymptotic boundary conditions [Eq. (6) of Ref. 1] were satisfied by successive approximations to an assumed velocity profile. In view of the computing power available to the authors [CDC 6600] it is unfortunate that they chose to use such a relatively crude integration scheme. Table 1 is presented in support of this comment. Here a comparison of the eigenvalues for the zero mass transfer separation solutions calculated by Fox and Saland,[1] Cohen and Reshotko[2] and by Rogers[3] are shown. The Cohen and Reshotko results were obtained using both the method of successive approximation with a variable step-size trapezoidal rule and by forward integration with a five-point integration formula (see Ref. 4, Appendix B). Cohen and Reshotko consider their results to be accurate to within ± 0.0002. Rogers[3] used a fourth-order Runge-Kutta integration scheme with a fixed step-size of 0.01. The asymptotic boundary conditions were satisfied using a Nachtsheim-Swigert iteration scheme.[5] Further, Rogers[3] performed comparative calculations with a step-size of 0.001 and on this basis conservatively estimated the accuracy of his results as $\pm 5 \times 10^{-5}$. Examination of Table 1 reveals that in general the results given in Ref. 1 may be considered accurate only to the third decimal place. In particular, the result for $g_w = 1.0$ which corresponds to the separation solution of the classical Falkner-Skan equation has been determined by a number of authors. The present author has calculated this result in single precision to seven significant figures which yields $\beta = -0.1988376$ with $\eta_{\max} = 9.0$. Nachtsheim and Swigert,[5] using an Adams-Moulton integration scheme with a fixed step-size of 0.0002, calculated $\beta = -0.198837682$ in double precision with $\eta_{\max} = 16.0$. Fox and Saland's calculations yield $\beta = -0.19901$.

To further support this comment the unit Prandtl number separation solutions for $f_w = 1.0$ are shown in Table 2. These solutions were calculated using a fourth-order Runge-Kutta integration scheme with a fixed step-size of 0.01. Using a Nachtsheim-Swigert iteration scheme the asymptotic outer boundary conditions were considered to be satisfied if $|1 - f'|_{\eta=\eta_{\max}}$ and $|1 - g|_{\eta=\eta_{\max}}$ were simultaneously less than 5×10^{-7} and if $|f''|_{\eta=\eta_{\max}}$ and $|g'|_{\eta=\eta_{\max}}$ were less than 5×10^{-6} and decreasing. Comparison of these results with those presented in Table 2 of Ref. 1 again indicates that the approximate accuracy of the results in Ref. 1 is in the third decimal place.

In addition, Fox and Saland[1] indicate that reverse flow solutions cannot be obtained by standard numerical techniques. The present author[6] has fully documented that the discontinuous solution branches discussed by Libby and Liu[7] are readily obtained by standard numerical techniques, namely the fourth-order Runge-Kutta Nachtsheim-Swigert technique discussed above. Further, it appears[6] that this technique suppresses the algebraically decaying solutions.

In order to further demonstrate that the lower branch and/or reverse flow solutions can be obtained by standard numerical techniques, the boundary value problem associated with similar solutions of the compressible laminar boundary

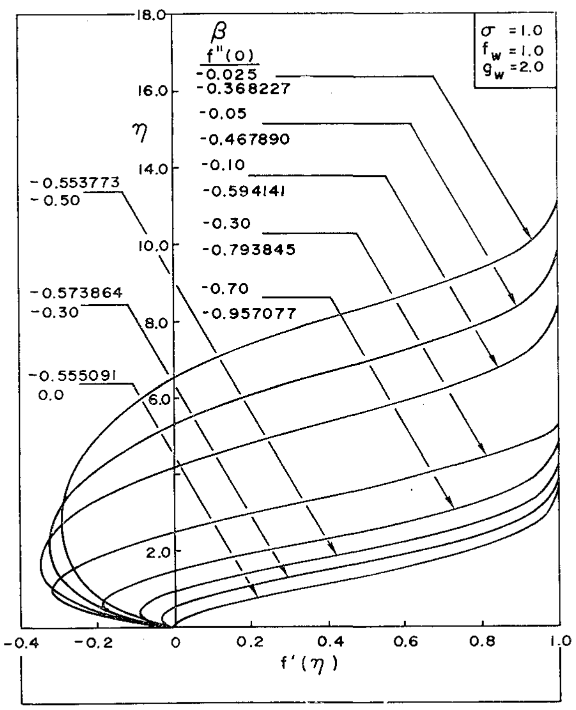

Fig. 1 Nondimensional velocity profiles $\sigma = 1.0, f_w = 1.0, g_w = 2.0$.

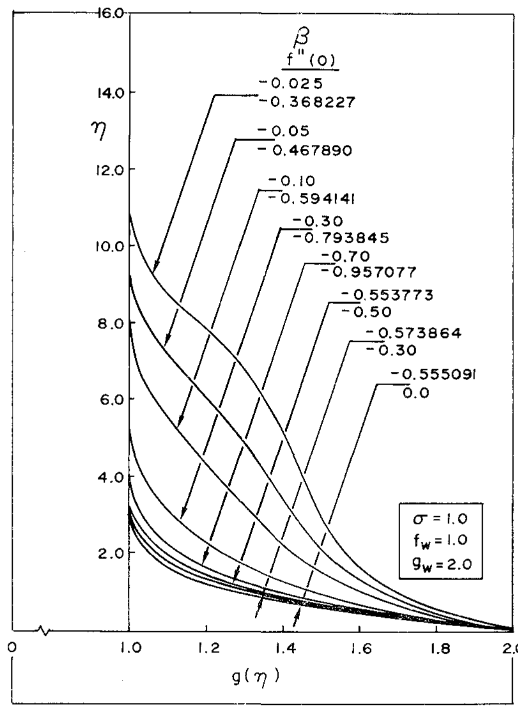

Fig. 2 Nondimensional enthalpy profiles $\sigma = 1.0, f_w = 1.0, g_w = 2.0$.

Table 2 Separation solutions of the similar compressible laminar boundary-layer equations with mass transfer $\sigma = 1.0, f''(0) = 0, f(0) = 1.0$

g_w	$-\hat{\beta}$	$g'(0)$	$\delta_{tro}{}^*$	θ_{tr}
0	0.822459	1.075035	1.321262	0.488257
0.2	0.816288	0.871091	1.334238	0.485111
0.6	0.774632	0.444047	1.427348	0.468856
1.0	0.712041	0.0	1.586740	0.450854
2.0	0.555091	−1.144072	2.138149	0.420017

layer with mass transfer is integrated for the particular case of unit Prandtl number ($\sigma = 1$), a surface to stagnation enthalpy ratio $g_w = 2.0$, and a similar mass transfer parameter $f_w = 1.0$, i.e.,

$$f''' + ff'' + \hat{\beta}(g - f'^2) = 0 \tag{1}$$

$$g'' + fg' = 0 \tag{2}$$

with boundary conditions

$$f(0) = f_w = 1.0 \tag{3a}$$

$$f'(0) = 0 \tag{3b}$$

$$g(0) = g_w = 2.0 \tag{3c}$$

$$f'(\eta \rightarrow \infty) = g(\eta \rightarrow \infty) \rightarrow 1\dagger \tag{3d}$$

These solutions were obtained using the fourth-order Runge-Kutta Nachtsheim-Swigert technique discussed above. The results which are characteristic of reverse flow solutions are given in Table 3 and Figs. 1 and 2.

Table 3 Reverse flow solutions of the similar compressible laminar boundary-layer equations with mass transfer $\sigma = 1.0, g_w = 2.0, f(0) = 1.0$

$-\hat{\beta}$	$-f''(0)$	$-g'(0)$
0.555091	0	1.144072
0.566712	0.1	1.127216
0.573864	0.3	1.087995
0.553773	0.5	1.037012
0.482721	0.7	0.957077
0.4	0.782546	0.886807
0.3	0.793845	0.807186
0.2	0. 34598	0.722124
0.1	0.594141	0.615030
0.05	0.467890	0.536701
0.025	0.368227	0.478004

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† The notation is that of Reference 7.

Reply by Authors to David F. Rogers

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THE comments by David F. Rogers on Ref. 1 should be of value to those interested in extremely high degrees of accuracy. Our main purpose in developing the solutions presented there was to create the separation maps as displayed in Figs. 1, 2, 5, and 9 of Ref. 1. We were interested, predominantly, in obtaining the value of $\hat{\beta}$ at separation with accuracy adequate for engineering work and employment in local similarity analyses.

One observation made by Rogers should be clarified. With the numerical techniques employed, i.e., successive approximation, coupled with the specific initial guess chosen, $f'(\eta) = 1$,‡ no lower branch solutions can possibly be obtained. Careful inspection of the describing integral equations of Ref. 1 indicates that, with this initial guess, appearance of such solutions is precluded. This is not to say that with successive approximation the lower-branch solutions cannot be obtained; a different initial guess exhibiting lower-branch, reverse flow, behavior would probably be required.

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‡ Notation to the same as in Ref. 1.

Comment on "A Method for Extracting Aerodynamic Coefficients from Free-Flight Data"

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Introduction

IN a recent paper[1] Chapman and Kirk of NASA Ames Research Center developed a technique of finding the parameters of various differential equations from flight data. This method is an iterative one that does not require a closed-form solution of the differential equation and is conceptually much more attractive than older methods that employ fits of the data with combinations of damped sine waves. The Chapman-Kirk method does require lengthy calculations on a large computer, and thus it is desirable that it either can provide better results than the previously developed methods requiring less calculation or can be applied to differential equations that cannot be treated by other methods. In reviewing the paper, we note that only one of their four examples might be a realistic case of this class, i.e., a missile with nonlinear damping and static moments. A quasi-linear technique[2–4] can be used on this example, but Chapman and Kirk did not make a

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